

GRAVITATIONALLY BOUND MONOPOLES

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Abstract

We construct monopole solutions in $SU(2)$ Einstein-Yang-Mills-Higgs theory, carrying magnetic charge n . For vanishing and small Higgs selfcoupling, these multimonopole solutions are gravitationally bound. Their mass *per unit charge* is lower than the mass of the $n = 1$ monopole. For large Higgs selfcoupling only a repulsive phase exists.

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1 Introduction

Magnetic monopoles arise as topological defects in theories which undergo spontaneous symmetry breaking. In general, magnetic monopoles exist if the mapping of the vacuum manifold, associated with the symmetry breaking, onto the two-sphere is non-trivial. The existence of magnetic monopoles is consequently a generic prediction of grand unification. If magnetic monopoles were indeed present in the universe, they would have a host of astrophysical and cosmological consequences.

Here we consider magnetic monopoles [1] and multimonopoles [2, 3, 4, 5] in $SU(2)$ Yang-Mills-Higgs (YMH) theory, with the Higgs field in the triplet representation. These solutions are globally regular. Since their magnetic charge is proportional to their topological charge, the regular monopole and multimonopole solutions reside in topologically non-trivial sectors of the theory. In contrast, the monopole-antimonopole pair solutions [6, 7] are topologically trivial.

In the Bogomol'nyi-Prasad-Sommerfield (BPS) limit [8, 9], where the strength of the Higgs self-interaction potential vanishes, the mass of the monopole and multimonopole solutions saturates its lower bound, the Bogomol'nyi bound. In particular, the mass *per unit charge* of an $n > 1$ monopole is precisely equal to that of the $n = 1$ monopole. There is no interaction between the monopoles in the BPS limit, because the spatial components of the stress tensor vanish [10]. The Higgs field is massless and mediates a long range attractive force which exactly cancels the long range repulsive magnetic force of the $U(1)$ field [11, 12].

For finite Higgs selfcoupling, however, the Higgs field is massive and therefore decays exponentially. Consequently the long range magnetic field dominates at large distances, leading to the repulsion of like monopoles [13]. In particular, as verified numerically for $n = 2$ and $n = 3$ monopoles [4], the mass *per unit charge* of an $n > 1$ monopole is higher than the mass of the $n = 1$ monopole. Thus for finite Higgs selfcoupling there is only a repulsive phase between like monopoles.

Let us now consider the effect of gravity on the monopole solutions. When gravity is coupled to YMH theory, a branch of gravitating $n = 1$ monopole solutions emerges smoothly from the flat space monopole solution [14]. The branch extends up to some maximal value of the gravitational strength, beyond which the size of the monopole core would be smaller than the Schwarzschild radius of the solution [14]. Along the branch with increasing gravitational strength, the mass of the gravitating monopole solutions decreases monotonically. Finally a degenerate horizon forms [14, 15, 16, 17].

Similarly, in the presence of gravity a branch of gravitating monopole-antimonopole pair (MAP) solutions emerges from the flat space MAP solution [18].

In this letter we investigate how gravity affects the static axially symmetric multimonopole solutions of $SU(2)$ YMH theory [2, 3, 4] and whether the inclusion of gravity allows for an attractive phase of like monopoles. For a given topological charge n ,

we find a branch of globally regular and asymptotically flat multimonopole solutions, emerging smoothly from the corresponding flat space solution, and extending up to some maximal value of the gravitational strength. Along the branch the mass of the solutions decreases monotonically. And, indeed, we find a region of parameter space, where the mass *per unit charge* of the gravitating multimonopole solutions is lower than the mass of the gravitating $n = 1$ monopole. Here the multimonopole solutions are gravitationally bound. Thus gravity allows for an attractive phase of like monopoles, not present in flat space.

2 Ansatz

We consider SU(2) EYMH theory with action

$$S = \int \left(\frac{R}{16\pi G} - \frac{1}{2e^2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) - \frac{1}{4} \text{Tr}(D_\mu \Phi D^\mu \Phi) - \frac{1}{4} \lambda \text{Tr}(\Phi^2 - \eta^2)^2 \right) \sqrt{-g} d^4x , \quad (1)$$

with Newton's constant G , Yang-Mills coupling constant e , and Higgs self-coupling constant λ .

In flat space SU(2) YMH possesses multimonopole solutions with axial symmetry for any topological number [2]. For topological number $n \geq 3$, solutions with less symmetry exist [5]. Here we extend the axially symmetric multimonopole solutions to curved space. We choose isotropic coordinates with metric [19]

$$ds^2 = -f dt^2 + \frac{m}{f} (dr^2 + r^2 d\theta^2) + \frac{l}{f} r^2 \sin^2 \theta d\varphi^2 , \quad (2)$$

where f , m and l are only functions of r and θ . The ansatz for the purely magnetic gauge field is [3, 19]

$$A_\mu dx^\mu = \frac{1}{2er} \left[\tau_\phi^n (H_1 dr + (1 - H_2) r d\theta) - n (\tau_r^n H_3 + \tau_\theta^n (1 - H_4)) r \sin \theta d\phi \right] , \quad (3)$$

and for the Higgs field the ansatz is [3, 4]

$$\Phi = (\Phi_1 \tau_r^n + \Phi_2 \tau_\theta^n) . \quad (4)$$

The symbols τ_r^n , τ_θ^n and τ_ϕ^n denote the dot products of the cartesian vector of Pauli matrices, $\vec{\tau} = (\tau_x, \tau_y, \tau_z)$, with the spatial unit vectors

$$\begin{aligned} \vec{e}_r^n &= (\sin \theta \cos n\phi, \sin \theta \sin n\phi, \cos \theta) , \\ \vec{e}_\theta^n &= (\cos \theta \cos n\phi, \cos \theta \sin n\phi, -\sin \theta) , \\ \vec{e}_\phi^n &= (-\sin n\phi, \cos n\phi, 0) , \end{aligned} \quad (5)$$

respectively. The winding number n represents the topological charge of the solutions. The four gauge field functions H_i and the two Higgs field function Φ_i depend only on the coordinates r and θ . For $n = 1$ and $H_1 = H_3 = \Phi_2 = 0$, $H_2 = H_4 = K(r)$ and $\Phi_1 = H(r)$, the spherically symmetric ansatz of ref. [14] is recovered.

We fix the residual gauge degree of freedom [3, 19] by choosing the gauge condition $r\partial_r H_1 - \partial_\theta H_2 = 0$ [19].

For asymptotically flat magnetically charged solutions the boundary conditions at infinity are

$$f = m = l = 1, \quad H_1 = H_2 = H_3 = H_4 = 0, \quad \Phi_1 = \eta, \Phi_2 = 0. \quad (6)$$

Requiring the solutions to be regular at the origin ($r = 0$) leads to the boundary conditions

$$\partial_r f = \partial_r m = \partial_r l = 0, \quad H_1 = H_3 = 0, \quad H_2 = H_4 = 1, \quad \Phi_1 = \Phi_2 = 0. \quad (7)$$

The boundary conditions along the ρ - and z -axis are determined by the symmetries of the solutions. On both axes the functions H_1, H_3 and Φ_2 and the derivatives $\partial_\theta f, \partial_\theta m, \partial_\theta l, \partial_\theta H_2, \partial_\theta H_4, \partial_\theta \Phi_1$ have to vanish.

Introducing the dimensionless coordinate $x = r\eta e$ and the Higgs field $\phi = \Phi/\eta$, the equations depend only on two coupling constants, α and β ,

$$\alpha^2 = 4\pi G\eta^2, \quad \beta^2 = \frac{\lambda}{e^2}. \quad (8)$$

The mass M of the multimonopole solutions can be obtained directly from the total energy-momentum “tensor” $\tau^{\mu\nu}$ of matter and gravitation, $M = \int \tau^{00} d^3r$ [20], or equivalently from $M = - \int (2T_0^0 - T_\mu^\mu) \sqrt{-g} dr d\theta d\phi$, yielding the dimensionless mass $\mu = \frac{e}{4\pi\eta} M$.

3 Results

Subject to the above boundary conditions, we solve the equations numerically [21]. We start from the flat space (multi)monopole solutions [4] and increase α . Then branches of gravitating (multi)monopole solutions extend up to maximal values of α , $\alpha_{\max}^{(n)}(\beta)$, which increase with increasing topological charge and decrease with increasing strength of the Higgs selfcoupling.

As the globally regular (multi)monopole solutions approach the critical value of α , a degenerate horizon starts to form [14, 15, 16]. Indeed, the exterior of the critical solution corresponds to the exterior of an extremal Reissner-Nordström (RN) black hole with magnetic charge n [14, 15, 17].

In Fig. 1 we show the metric function f of the gravitating $n = 2$ multimono-
poles in the BPS limit for several values of α . With increasing α the function f decreases
monotonically and approaches the metric function $f^{\text{RN}} = \left(\frac{x/\alpha}{n+x/\alpha}\right)^2$ of the extremal RN
solution with magnetic charge $n = 2$. At $\alpha = 1.5$, its value at the origin is already
very small, close to the value $f^{\text{RN}}(0) = 0$, representing the degenerate horizon of the
extremal RN solution. Also, the angle-dependence of the metric function is seen to
diminish and disappear, as the critical value of α is approached. Likewise, the Higgs
field function ϕ_1 , shown in Fig. 2, approaches the value $\phi_1^{\text{RN}} = 1$ of the RN solution [22].
The Higgs field function ϕ_2 and the gauge field functions approach their respective RN
values as well. Satisfying Israel's theorem, the RN solution is spherically symmetric.

The mass *per unit charge* of the (multi)monopole solutions decreases with increasing
 α . In the BPS limit, for $\alpha = 0$ the mass *per unit charge* is precisely equal to the mass
of the $n = 1$ monopole. For $\alpha > 0$, however, we observe that the mass *per unit charge*
of the multimono-
poles is smaller than the mass of the $n = 1$ monopole. In particular,
the mass *per unit charge* decreases with increasing n . The mass *per unit charge* of
 $n = 2$ and 3 multimono-
poles in the BPS limit is shown in Fig. 3. Thus in the BPS
limit, there is an attractive phase between like mono-
poles, not present in flat space.
Moreover, multimono-
poles exist for gravitational coupling strength, too large for $n = 1$
mono-
poles to exist.

For finite Higgs selfcoupling, the flat space multimono-
poles have higher mass *per unit charge* than the $n = 1$ monopole, allowing only for a repulsive phase between
like mono-
poles. By continuity, this repulsive phase persists in the presence of gravity
for small values of α , but it can give way to an attractive phase for larger values
of α . Thus the repulsion between like mono-
poles can be overcome for small Higgs
selfcoupling by sufficiently strong gravitational attraction. At the equilibrium value α_{eq} ,
multimono-
pole mass *per unit charge* and monopole mass equal one another. We show
the equilibrium value α_{eq} in Fig. 4. α_{eq} increases with increasing Higgs selfcoupling,
yielding a decreasing region in parameter space for the attractive phase. Finally, for
large Higgs selfcoupling, only a repulsive phase is left.

While singly charged monopole solutions are stable, stability of the static axially
symmetric multimono-
pole solutions is not obvious. We conjecture, that the $n = 2$
multimono-
pole solutions are stable, as long as their mass *per unit charge* is lower
than the mass of the $n = 1$ monopole. For topological number $n \geq 3$, however,
solutions with only discrete symmetry exist in flat space [5], which, by continuity,
should also be present in curved space (at least for small gravitational strength). For
a given topological number $n > 2$, such multimono-
pole solutions without rotational
symmetry may possess a lower mass than the corresponding axially symmetric solutions.
The axially symmetric solutions may therefore not represent global minima in their
respective topological sectors, even if their mass *per unit charge* is lower than the mass
of the $n = 1$ monopole.

4 Black Holes

Besides embedded abelian black hole solutions, like e.g. the magnetically charged RN solutions encountered above, SU(2) EYMH theory also possesses genuine non-abelian black hole solutions [14]. The SU(2) EYMH black hole solutions then are no longer uniquely determined by their mass and charge (for vanishing angular momentum). The non-abelian black hole solutions, considered as black holes within (multi)monopoles, therefore represent counterexamples to the “no-hair” theorem [14].

While static spherically symmetric ($n = 1$) EYMH black holes were studied in much detail [14], non-abelian black hole solutions with magnetic charge $n > 1$ were previously only obtained perturbatively [23]. Replacing the boundary conditions at the origin (7) by the appropriate set of boundary conditions at some horizon radius x_h [19, 21], we obtain static axially symmetric black hole solutions with magnetic charge $n > 1$ numerically [21]. These black hole solutions are asymptotically flat, as required by the boundary conditions at infinity (6), and they possess a regular deformed horizon [19, 21]. Being static and not spherically symmetric, these black hole solutions represent new counterexamples to Israel’s theorem. While previous (non-perturbative) counterexamples [19, 24] to Israel’s theorem are not classically stable, black holes within multimonopoles should provide classically stable counterexamples [23].

Finally we note, that the gravitational properties of monopole spacetimes near the black hole threshold, lend themselves to obtain new insight on the laws of black hole thermodynamics [25]. Also, insight concerning the cosmic censorship hypothesis might be obtained from the static black hole solutions without spherical symmetry [26].

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References

- [1] G. ‘t Hooft, Nucl. Phys. **B79** (1974) 276;
A. M. Polyakov, JETP Lett. **20** (1974) 194.
- [2] R. S. Ward, Commun. Math. Phys. **79** (1981) 317;
P. Forgacs, Z. Horvarth and L. Palla, Phys. Lett. **99B** (1981) 232;
P. Forgacs, Z. Horvarth and L. Palla, Nucl. Phys. **B192** (1981) 141;
M. K. Prasad, Commun. Math. Phys. **80** (1981) 137;
M. K. Prasad and P. Rossi, Phys. Rev. **D24** (1981) 2182;
E. Corrigan and P. Goddard, Commun. Math. Phys. **80** (1981) 575.
- [3] C. Rebbi and P. Rossi, Phys. Rev. **D22** (1980) 2010.
- [4] B. Kleihaus, J. Kunz and D. H. Tchraikian, Mod. Phys. Lett. **A13** (1998) 2523.

- [5] see P. M. Sutcliffe, Int. J. Mod. Phys. **A12** (1997) 4663.
- [6] C. H. Taubes, Commun. Math. Phys. **86** (1982) 257;
C. H. Taubes, Commun. Math. Phys. **86** (1982) 299.
- [7] Bernhard Rüber, Thesis, University of Bonn 1985;
B. Kleihaus and J. Kunz, Phys. Rev. **D61** (2000) 025003.
- [8] E. B. Bogomol'nyi, Yad. Fiz. **24** (1976) 499 [Sov. J. Nucl. Phys. **24** (1976) 861].
- [9] M. K. Prasad and C.M. Sommerfield, Phys. Rev. Lett. **35** (1975) 760.
- [10] see A. Jaffe and C. J. Taubes, Monopoles and Vortices, Birkhäuser, Basel, 1980.
- [11] N. S. Manton, Nucl. Phys. **B126** (1977) 525.
- [12] W. Nahm, Phys. Lett. **B79** (1978) 426;
W. Nahm, Phys. Lett. **B85** (1979) 373.
- [13] J. N. Goldberg, P. S. Jang, S. Y. Park and K. C. Wali, Phys. Rev. **D18** (1978) 542.
- [14] K. Lee, V.P. Nair and E.J. Weinberg, Phys. Rev. **D45** (1992) 2751;
P. Breitenlohner, P. Forgacs and D. Maison, Nucl. Phys. **B383** (1992) 357;
P. Breitenlohner, P. Forgacs and D. Maison, Nucl. Phys. **B442** (1995) 126.
- [15] A. Lue and E.J. Weinberg, Phys. Rev. **D60** (1999) 084025.
- [16] For $n = 1$ monopoles for zero and very small Higgs self-coupling a short second branch extends backwards from the maximal value of α , α_{\max} , to the critical value of α , α_{cr} , where the extremal RN solution is reached.
- [17] For $n = 1$ monopoles for large Higgs self-coupling the exterior solution is not of RN-type [15].
- [18] B. Kleihaus and J. Kunz, Phys. Rev. Lett. **85** (2000) 2430.
- [19] B. Kleihaus and J. Kunz, Phys. Rev. Lett. **78**, 2527 (1997);
B. Kleihaus and J. Kunz, Phys. Rev. Lett. **79**, 1595 (1997);
B. Kleihaus and J. Kunz, Phys. Rev. **D57**, 834 (1998);
B. Kleihaus and J. Kunz, Phys. Rev. **D57**, 6138 (1998).
- [20] S. Weinberg, Gravitation and Cosmology (Wiley, New York, 1972)
- [21] B. Hartmann, B. Kleihaus, and J. Kunz, in preparation.

- [22] The gauge invariant modulus of the Higgs field is very close to the Higgs field function ϕ_1 .
- [23] S. A. Ridgway and E. J. Weinberg, Phys. Rev. **D52** (1995) 3440.
- [24] B. Kleihaus and J. Kunz, Non-abelian black holes with magnetic dipole hair, Phys. Lett. **B494** (2000) 130.
- [25] A. Lue and E.J. Weinberg, Phys. Rev. **D61** (2000) 124003;
A. Lue and E.J. Weinberg, Monopoles and the emergence of black hole entropy, gr-qc/0004001.
- [26] We thank the referee for pointing this out along with reference F. I. Cooperstock, S. Jhingan, P. S. Joshi and T. P. Singh, Class. Quantum Grav. **14** (1997) 2195.

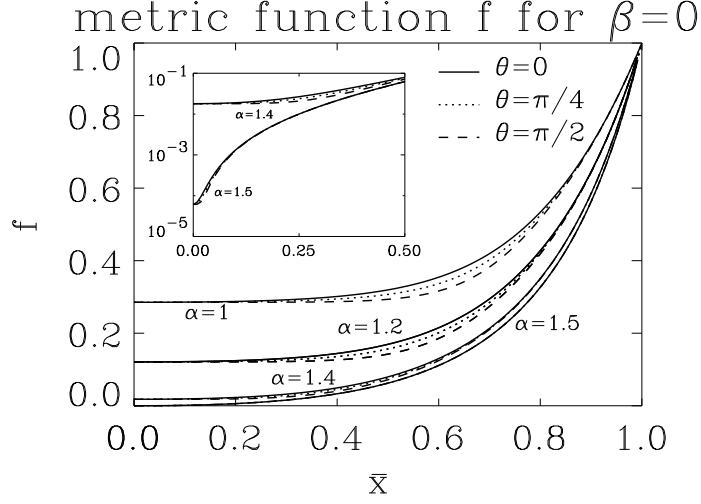


Figure 1: The metric function f is shown as a function of the compactified dimensionless radial coordinate $\bar{x} = x/(1+x)$ for three values of the angle θ for the $n=2$ multimonopole solutions in the BPS limit, for a sequence of values of α , approaching the critical value α_{cr} .

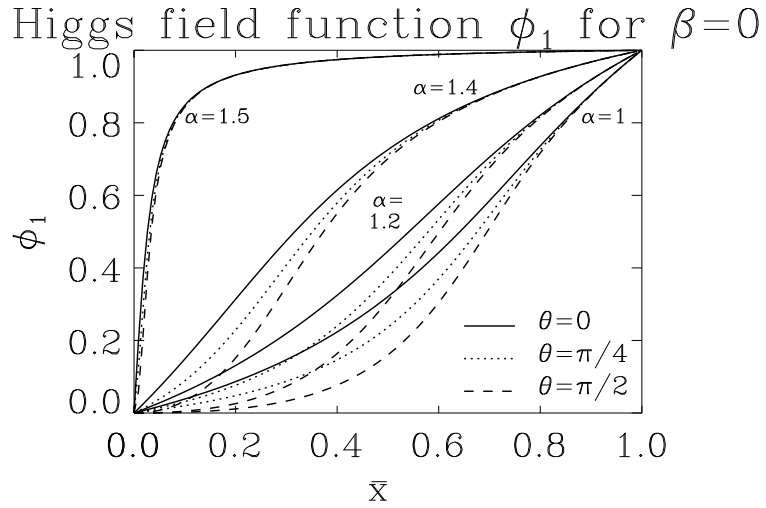


Figure 2: Same as Fig. 1 for the Higgs field function ϕ_1 .

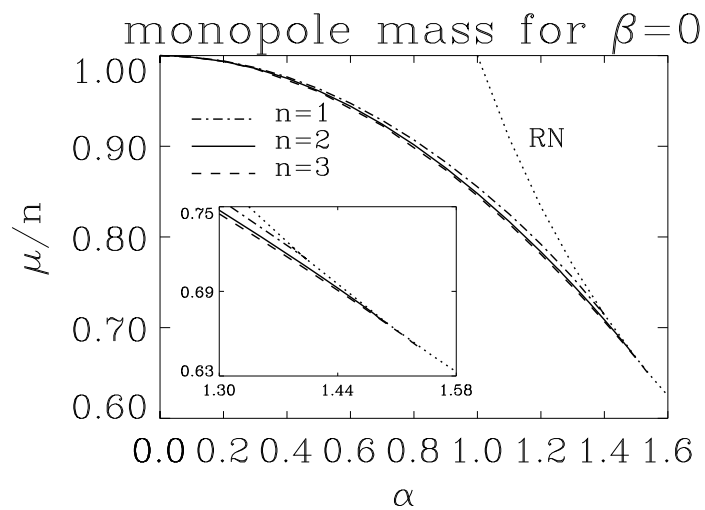


Figure 3: The mass *per unit charge* in units of $4\pi\eta/e$ is shown as a function of α for the $n = 2$ and 3 multimonopoles in the BPS limit. For comparison the mass of the monopole and the mass of the extremal RN solutions are also shown.

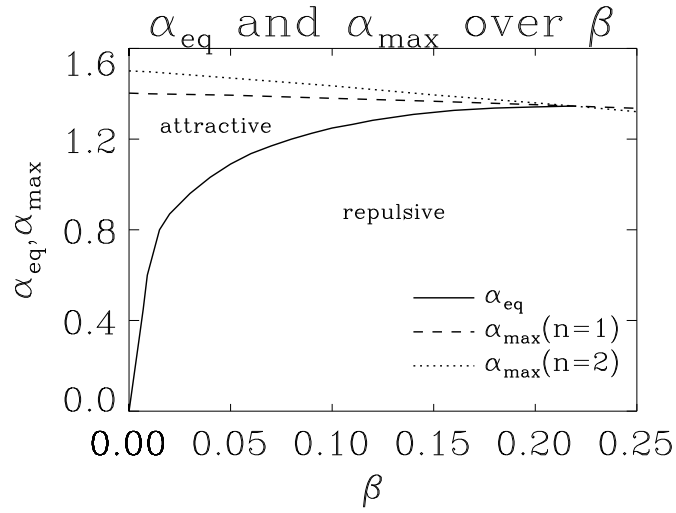


Figure 4: The equilibrium value of α , α_{eq} , where the multimono-*pole mass per unit charge* and monopole mass equal one another, is shown as a function of β for the $n = 2$ multimono-*poles*. Also shown are the maximal values α_{max} for the $n = 1$ and 2 (multi)mono-*poles*.